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C. Bernardini and C. Pellegrini: NON PERTURBATIVE SOLUTION OF SMALL ANGLE SCATTERING PROBLEMS (MOLIERE'S APPROXIMATION).

1 - We want to describe a simple procedure which leads to a well known result by Molière(1) concerning the small angle scattering of a particle by the potential due to an extended source. Our procedure is somewhat more general and, at the same time, more intuitive; it seems to us that this is an occasion to recall attention on this useful approximation, which has applications to a wide variety of high energy problems and is often ignored in the literature.

The basis of the method is the assumption that the Fourier transform (over space coordinates) of the potential falls down to zero rapidly enough at large wave numbers. That is, there must be a cutoff (eventually related to the size of the scatterer) at a characteristic momentum much smaller than that of the incoming or the outgoing particle.

Special care must be given in handling the case of long range potentials because of the well-known phase divergence(2) of the Coulomb problem. We give here a solution in a form well suited for both short and long range sources: the use of coulomb wave functions for this last case could perhaps be more appropriate but we did not follow this approach in detail.

2 - We start from a wave equation in the form

$$i\dot{\psi} = (H_o + V)\psi \tag{1}$$

and assume as a basis the plane wave solutions of the free particle equation

$$i\,\dot{\psi}_f = H_o\,\psi_f$$

$$\psi_f(\vec{x}, t) = \chi(\vec{p})\, e^{-iE(p)t}\, e^{i\vec{p}\cdot\vec{x}} \tag{2}$$

Here $\chi(\vec{p})$ can be a normalization constant or a free-particle Dirac spinor according to the explicit form of wave equation (1). The units are such that $\hbar = c = 1$.

Putting now

$$\psi(\vec{x}, t) = \int c(\vec{p}, t)\, \chi(\vec{p})\, e^{-iE(p)t}\, e^{i\vec{p}\cdot\vec{x}}\, d^3p$$

eq. (1) transforms into

$$i\,\dot{c}(\vec{p}, t) = \int d^3p'\, U(\vec{p}, \vec{p}')\, C(\vec{p}', t)\, e^{i[E(p) - E(p')]t} \tag{3}$$

where the kernel of the integral term is defined by

$$U(\vec{p}, \vec{p}') = \chi^*(\vec{p})\, V(\vec{p} - \vec{p}')\, \chi(\vec{p}')$$

and

$$V(\vec{k}) = \frac{1}{(2\pi)^3} \int e^{i\vec{k}\cdot\vec{x}}\, V(\vec{x})\, d^3x$$

we use the normalization to one particle per unit volume so that

$$\chi^*(\vec{p})\, \chi(\vec{p}) = 1$$

3 - We assume in the following that

α - $V(\vec{k})$ falls down rapidly to zero for $k > k_c$ (k_c will be called the cutoff momentum)

β - the cutoff momentum k_c is much smaller than the initial or final momenta of the scattered particle.

It can be shown that, because of energy conservation, $\chi(\vec{p}')$ differs from $\chi(\vec{p})$ by terms of order $\leq (k_c/p)^2$ in the relevant part of the integration volume in (3); therefore we will approximate

$$U(\vec{p}, \vec{p}') \simeq V(\vec{p} - \vec{p}')$$

This approximation is not quite necessary but simplifies very much the calculations; as a consequence of it, spin effects are ignored (but could have been included as well).

Now, the main point of the method is the use of statements α) and β) to write

$$E(p) - E(p') \simeq (\vec{p} - \vec{p}') \cdot \frac{\vec{p}}{E(p)} = (\vec{p} - \vec{p}') \cdot \vec{v}$$

in the time-dependent exponent in eq. (3).

Furthermore, restricting ourselves to small angle scattering, we write $\vec{v} \simeq \vec{v}_o$, the initial velocity of the particle. Eq. (3) eventually reduces to the simpler form

$$i\,\dot{c}(\vec{p}, t) = \int d^3k\, V(\vec{k})\, c(\vec{p} - \vec{k}, t)\, e^{i\,\vec{k}\cdot\vec{v}_o t} \tag{4}$$

where the r. h. s. has the structure of a convolution integral.

4 - We solve now eq. (4) assuming that the particle has momentum $\vec{p}_o$ at $t = -\infty$. This means that:

$$c(\vec{p}, -\infty) = (2\pi)^3\, \delta(\vec{p} - \vec{p}_o) \tag{5}$$

By using the faltung theorem the solution of eq. (4) satisfying (5) can be written in the form

(6) $$c(\vec{p}, t) = \int d^3x\, e^{i(\vec{p} - \vec{p}_o).\vec{x}}\; e^{-i\int_{-\infty}^{t} V(\vec{x} - \vec{v}_o t')\, dt'}$$

We want to calculate the scattering cross-section and therefore we are interested in the limit

$$\lim_{t \to \infty} c(\vec{p}, t) = c_{\infty}(\vec{p})$$

First consider the exponent of the second exp factor in (6) in this limit. By using

$$\lim_{t \to \infty} \int_{-\infty}^{t} e^{i\alpha t'}\, dt' = 2\pi(\delta)(\alpha)$$

it is readily found that

$$\lim_{t \to \infty} \int_{o}^{t} V(\vec{x} - \vec{v}_o t')\, dt' = 2\pi \int d^3k\, e^{-i\vec{k}.\vec{x}}\, V(\vec{k})\, \delta(\vec{k}.\vec{v}_o)$$

Let us call $\vec{A}_{\perp}$ the component of any vector $\vec{A}$ in the plane orthogonal to the direction of $\vec{v}_o$. Then

(7) $$\lim_{t \to \infty} \int_{-\infty}^{t} V(\vec{x} - \vec{v}_o t')\, dt' = \frac{2\pi}{v_o} \int d^2k_{\perp}\, e^{-i\vec{k}_{\perp}.\vec{x}_{\perp}}\, V(\vec{k}_{\perp}) = \Omega(\vec{x}_{\perp}) \text{ say}$$

Here $d^2k_{\perp}$ is a surface element on the plane $\vec{k}.\vec{v}_o = 0$ and the integration is performed on this plane. It follows that

(8) $$C_{\infty}(\vec{p}) = 2\pi v_o\, \delta\left[\vec{v}_o.(\vec{p} - \vec{p}_o)\right] \int d^2x_{\perp}\, e^{i(\vec{p} - \vec{p}_o).\vec{x}_{\perp} - i\Omega(\vec{x}_{\perp})}$$

The δ function appearing in this formula guarantees the conservation of the energy; in fact, the main approximation we used is equiva to put

$$E(\vec{p}) - E(\vec{p}_o) \simeq \vec{v}_o \cdot (\vec{p} - \vec{p}_o)$$

After subtraction of the incoming wave, the scattered amplitude is given by

(9) $$C_{sc}(\vec{q}) = 2\pi v_o \delta(\vec{v}_o \cdot \vec{q}) \int d^2x_\perp \, e^{i\vec{q}_\perp \cdot \vec{x}_\perp} \left[e^{-i\Omega(\vec{x}_\perp)} - 1 \right]$$

where $\vec{q} = \vec{p} - \vec{p}_o$ is the transferred momentum related to the scattering angle θ by

$$q = 2 \, p_o \sin \theta/2$$

The cross section can now be evaluated and is

(10) $$\frac{d\sigma}{d\Omega} = \frac{1}{4\pi^2} \, p_o^2 \left| \int d^2x_\perp \left[1 - e^{-i\Omega(\vec{x}_\perp)} \right] e^{i\vec{q}_\perp \cdot \vec{x}_\perp} \right|^2$$

This is Moliere's main result. The connection with the Born approximation is easily seen: when

(11) $$1 - e^{-i\Omega(x_\perp)} \simeq i\Omega(x_\perp)$$

then

$$\int d^2x_\perp \left[1 - e^{-i\Omega(\vec{x}_\perp)} \right] e^{i\vec{q}_\perp \cdot \vec{x}_\perp} \simeq \frac{(2\pi)^3}{v_o} V(\vec{q}_\perp)$$

and the cross section becomes simply

(12) $$\frac{d\sigma}{d\Omega} \simeq (2\pi)^4 \frac{p_o^2}{v_o^2} \left| V(\vec{q}_\perp) \right|^2$$

Formula (10) is valid for a wider class of potentials (not satisfying (11)) than (12) but only at small scattering angles.

5 - Formula (10) can be used when

$$V(k) \longrightarrow \frac{1}{k^2}\ , \quad k \longrightarrow o \tag{13}$$

that is in the long-range case, following a special procedure to remove phase-divergences. Since, in this case, we are not interested in the (infinite) forward scattering cross section, we can rewrite

$$\frac{d\sigma}{d\Omega} = \frac{p_o^2}{4\pi^2} \left| \int d^2x_\perp \; e^{-i\,\Omega(\vec{x}_\perp)} \; e^{i\,\vec{q}_\perp \cdot \vec{x}_\perp} \right|^2 \tag{14}$$

$\Omega(\vec{x}_\perp)$ diverges like $\log \mu$ when μ, the inverse range of the forces, goes to zero. To be more specific, we can define

$$V(\vec{k}) = \frac{Z\,e^2}{2\pi^2} \; \frac{F(\vec{k})}{k^2 + \mu^2}\ , \quad F(o) = 1 \tag{15}$$

where $F(\vec{k})$ is the form factor of the source of the Coulomb potential and Ze^2 recalls the typical strenght of the electron-nucleus scattering problem.

We can now relegate the divergence in an unrelevant phase factor by putting

$$\Omega(\vec{x}_\perp) = \Omega(o) - S(\vec{x}_\perp)$$

Since, in fact, $\Omega(\vec{x}_\perp)$ is real and $\Omega(o)$ contains the $\log \mu$ divergence, (14) is identical with

$$\frac{d\sigma}{d\Omega} = \frac{p_o^2}{4\pi^2} \left| \int d^2x_\perp \; e^{+iS(\vec{x}_\perp)} \; e^{i\,\vec{q}_\perp \cdot \vec{x}_\perp} \right|^2 \tag{14'}$$

and

$$S(\vec{x}_\perp) = \frac{2\pi}{v_o} \int d^2k_\perp \left(1 - e^{-i\vec{k}_\perp \cdot \vec{x}_\perp}\right) V(\vec{k}_\perp)$$

is a divergence-free integral.

The cutoff parameter k_c, for this special case, is contained in the form factor $F(\vec{k})$ (see (15)).

It can be easily shown that the exact result (for spinless particles) is obtained for point charges if the limit $k_c \to \infty$ is taken after the cross section (14') is computed. Once again a phase diverging like $\log k_c$ appears in the scattering amplitude; but it has no physical consequences so that one eventually gets

$$\lim_{\substack{k_c \to \infty \\ \mu \to 0}} \frac{d\sigma}{d\Omega} = \frac{4Z^2 e^4 p_o^2}{v_o^2 q^4}$$

that is the Rutherford cross section.

6 - Forward scattering means that the incident particle must not penetrate too deeply into the scattering center. Since the size of the center is roughly measured by the inverse cutoff momentum $1/k_c$, this is equivalent to say that the involved angular momenta are mainly $> p_o/k_c$. But, because of statement β), in § 3, $p/k_c >> 1$; that is the angular momenta contributing to this approximation are very large. The range of large angular momenta is the one in which classical features of the process are dominating; - Molière's derivation is just based on this point of view[1], allowing for the extension of the result to large angles.

Since it is known that Molière's formula satisfies the optical theorem[3] we can write down the total scattering cross section as

$$\sigma = 2 \,\mathrm{Re} \left\{ \int d^2 x_\perp \left[1 - e^{-i \Omega (\vec{x}_\perp)} \right] \right\} \tag{16}$$

Since $x_\perp$ plays the role of the collision parameter of the classical picture and $\Omega(\vec{x}_\perp)$ is the phase distortion of the plane wave of a particle whose classical trajectory has collision parameter $x_\perp$, formula (16) has a simple interpretation in terms of interference between incident and scattered waves.

We want to thank R. Gatto who reminded us of Molière's formula; we are also grateful to him for other valuable comments.

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(2) - R. H. Dalitz - Proc. Roy. Soc. (London) A 206, 509 (1951)

(3) - See for instance Wu and Ohmura - Quantum theory of scattering, section C § 5.